

On the Classical Limit of Spin Network Gravity: Two Conjectures

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Abstract

Estimates are given of the time scales which govern spreading of a coherent state wave packet. The estimates, based on dimensional analysis, suggest that spreading should be small for coherent states with average angular momentum of order 100 or larger. It is conjectured that in the classical limit, terms in the Hamiltonian which add a new vertex will be suppressed, compared to terms which modify the existing spin network without changing the number of vertices.

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I Introduction

In two previous papers, which I will refer to as papers 1 and 2, I constructed a set of coherent states for planar gravity waves [1, 2]. The earlier papers do not propose any Hamiltonian.

However, there are two (at least) questions about coherent states which cannot be answered without knowing the Hamiltonian, or at least knowing a little bit about the Hamiltonian. The two questions focus on the rate of spreading of the coherent state wave packet (section II), and the possible addition of non-classical, low spin vertices in the classical limit (section III). The little bit of knowledge comes from dimensional analysis. The answers given are necessarily tentative, and these questions should be asked again when more is known about the Hamiltonian.

At several points I use the following results, derived in appendices C through E of paper 2. The angles and angular momentum for the coherent state are Gaussian distributed, and the standard deviations of angles are order $1/\sqrt{\langle L \rangle}$, while the standard deviations of angular momenta are order $\sqrt{\langle L \rangle}$. $\langle L \rangle$ is the average, or peak value of the magnitude of angular momentum for the coherent state.

(Coherent states also contain a parameter t , and appendix C of paper 2 quotes a value of $1/\sqrt{t}$ for the standard deviation of L . However, appendix D shows t must be order $1/\langle L \rangle$ in order to minimize the size of small correction terms. Therefore the standard deviation of $1/\sqrt{t}$ is not an exception. All standard deviations are order $\sqrt{\langle L \rangle}$ or $1/\sqrt{\langle L \rangle}$.

II Wave Packet Spreading

In paper 2 I computed explicitly the matrix elements.

$$\langle u, \vec{p} \mid (\tilde{E} \text{ or } h) \mid u, \vec{p} \rangle,$$

where \tilde{E} and h are densitized inverse triad and holonomy, and the coherent states are labeled by a unitary matrix u and a vector \vec{p} . However, there are many other matrix elements of the form

$$\langle u', \vec{p}' \mid \tilde{E} \mid u, \vec{p} \rangle$$

which I did not consider. For $u' \approx u$ and $\vec{p}' \approx \vec{p}$, the overlap could be large. Do these matrix elements play a role? This is an apparent problem with dealing with an overcomplete set.

This problem is perhaps more apparent than real. In the coherent state approach to the classical limit, one works only with diagonal matrix elements, the expectation values

$$\langle u, \vec{p} | (\tilde{E} \text{ or } h) | u, \vec{p} \rangle.$$

One does not worry about matrix elements to other coherent states, but only about how rapidly the wave packet $|u, \vec{p}\rangle$ will spread. Provided the time scale governing the spreading is large, the expectation values can be used to make predictions.

A Spreading in the weak field limit

In the weak field limit the gravitational Hamiltonian decouples into a sum of oscillators. These are especially easy to treat using coherent state methods. We can expect something like

$$\langle \delta\tilde{E}(z, T=0) \rangle = A \cos(kz + \gamma) \quad (1)$$

for time $T=0$; and oscillator packets are known to follow the classical path exactly for $T>0$ [3]:

$$\langle \delta\tilde{E}(z, T) \rangle = A \cos(kz - \omega T + \gamma).$$

$\delta\tilde{E}$ is the fluctuation of \tilde{E} away from flat space.

Although the *average* value of oscillator displacement follows the classical path, this is not enough to guarantee classical behavior. The fluctuations around the average must also be small. For the usual oscillator these fluctuations are determined by $\delta\gamma$, the uncertainty in the phase γ introduced at eq. (1). This uncertainty is connected to the uncertainty in the number of quanta by $\delta N \delta\gamma \sim 1$, which gives $\delta\gamma \sim 1/\sqrt{N}$, since N has standard deviation \sqrt{N} . In the LQG case presumably γ will be a function of the dimensionless angles (α, β) used to define the unitary matrix u , as well as the angles defining the unit vector \hat{p} . (γ could also be a function of a dimensionless ratio of angular momenta, (average Z component) over $\langle L \rangle$; but this ratio depends on the angles already listed.) The standard deviations for fluctuations in the angles are order $1/\sqrt{\langle L \rangle}$. $\langle L \rangle$ therefore replaces N ; and for L of order 100 or so, the fluctuations around the classical path should be small.

B Spreading in the strong field limit

Once the self-interaction of the gravitational packet is included, it becomes much harder to estimate the rate of spreading. Consider two familiar quantum mechanical examples, the free particle and the simple harmonic oscillator. The spreading of the free particle wave packet is governed by the time scale

$$T_x = m(\sigma_x)^2/\hbar, \quad (2)$$

where m is the mass and σ_x is the standard deviation of the $T = 0$ Gaussian packet in configuration space. At the other extreme, the Gaussian packet for the simple harmonic oscillator does not spread at all [3]. Evidently the rate of spreading is highly sensitive to the details of the energy spectrum [4].

The planar case, as well as the general $SU(2)$ case, can be given an asymptotic region, so can have a Hamiltonian [5, 6]; and it makes sense to talk about energies E . If the energy eigenvalues are evenly spaced, resembling those of the oscillator, then the likelihood of spreading should be small.

The Hamiltonian is a surface term. I will not try to construct this surface term in spin network theory, but rather will estimate it using dimensional analysis. The surface term can be rewritten as a density. For example for the planar case,

$$E = J|_{-\infty}^{+\infty} = \int \partial_z J dz,$$

and the $\partial_z J$ can in turn be expressed in terms of other fields using the classical equations of motion. One ends up with a function of the \tilde{E} and h . I assume this function resembles the typical terms in the Euclidean Hamiltonian.

$$E \sim \epsilon^{ijk} \text{Tr}(h_{ij} h_k [h_k^{-1}, V])/\kappa^2. \quad (3)$$

(I could also use the terms in the Hamiltonian which are proportional to the square of the extrinsic curvature; the order of magnitude estimates would be the same.) I need to estimate the dependence of this expression on angular momentum L . The volume V contains three \tilde{E} operators, integrated over area, with area eigenvalues of order $L\kappa$. Therefore volume V should be order $(L\kappa)^{3/2}$. However, the commutator of the volume with holonomy takes the derivative of

V with respect to L; see the discussion of the commutator in paper 2. Therefore $h_k [h_k^{-1}, V]$ is order $\sqrt{L}(\kappa)^{3/2}$. The remaining holonomies in the Hamiltonian give a result of order unity when acting on a state. Therefore the energy grows as the square root of L.

$$E \sim \sqrt{(L\kappa)}/\kappa. \quad (4)$$

This resembles the classical expression for the gravitational Hamiltonian, integral of (curvature) $d^3x/\kappa \simeq (\text{large length})/\kappa$, except that the large length has been replaced by the square root of an area eigenvalue. Also, the mass and event horizon area of a black hole are connected by a relation of similar form, mass \propto square root of area.

For minimal spreading, the spacing between energy levels should be as constant as possible, resembling the spacing between levels of the usual oscillator [4]. From eq. (4) the spacing is order

$$\delta E \sim \hbar c \delta L / \sqrt{L\kappa} \quad (5)$$

I have restored factors of \hbar and c , and given κ the dimensions of a length. For $\delta L = 1$, define a frequency ω by

$$\hbar\omega := \delta E(\delta L = 1) \sim \hbar c / \sqrt{L\kappa}. \quad (6)$$

At first glance this result is not what we want: the quantity ω is not a constant, independent of L. However, ω does not need to be a constant everywhere; it must be an approximate constant for the range of L values contained in a coherent state. Over this range, the fractional change in ω is of order

$$\delta\omega/\omega = -\delta L/(2L), \quad (7)$$

where now δL is the range of L values in the coherent state. Those L values are Gaussian distributed with a standard deviation $\sqrt{1/t}$. Using this standard deviation to estimate δL , I get

$$\delta\omega/\omega = -1/(2\sqrt{t}L). \quad (8)$$

As mentioned in the introduction, the parameter t must be of order $t \sim 1/ \langle L \rangle$ in order to minimize small correction terms. Inserting this value into eq. (8), and replacing L by $\langle L \rangle$, I get

$$\delta\omega/\omega \sim -1/(\sqrt{\langle L \rangle}). \quad (9)$$

Even in the strong field case, a value $\langle L \rangle \geq 100$ should be enough to drive $\delta\omega$ to zero and prevent spreading.

Eq. (8) is yet another reason why we cannot make t arbitrarily small: the spacing between levels would no longer be uniform over the packet, leading to unacceptable spreading. It is perhaps relevant that t plays the role of a standard deviation in spin network theory, and the time scale T_x for spreading of the free particle packet is also sensitive to a standard deviation, σ_x ; see eq. (2).

III New Vertices in the Classical Limit

It is not immediately clear that the action of the Hamiltonian, in the classical limit, involves only grasps at vertices with very high spin. The Hamiltonian probably contains terms which add a new vertex to the spin network, a vertex which therefore involves the minimum spin, spin 1/2. Consider the Euclidean term in the Hamiltonian for concreteness. (Since the remaining terms follow from commutators involving the Euclidean term, those terms will inherit the properties of the Euclidean term.) The spin network version of this term contains the operators shown in eq. (3). Consider, for example, the term $h_{ij} = h_{az}$ in this equation. This is a line integral of the holonomy along a contour with sides parallel to directions a ($= x$ or y) and z . To visualize the contour, fold a rectangular sheet of paper into a cylinder, until the two opposite edges almost touch. The contour is given by the boundary of the sheet, the two almost touching edges plus the two circular ends. Align the two almost touching edges with the z direction of the spin network; align the circular ends with the transverse a direction. (The contours in the transverse directions are always closed loops.) The contour is not completely closed. It must be opened, at one corner of the original sheet of paper, in order to insert the $[V, h]$ commutator.

The above description is not enough to define the Hamiltonian. If the spin network contains vertices v_1, v_2, \dots, v_n arranged along the z axis, one must specify how the holonomic contour is positioned with respect to these vertices. Evidently one of the circular ends, the one containing the $[V, h]$ commutator, must coincide with one of the vertices, say v_1 ; otherwise the volume operator in $[V, h]$ will give zero. As for the other circular end, the two simplest possibilities are: that end coincides with the nearest neighbor, vertex v_2 ; or,

that end creates a new vertex with an associated spin 1/2 loop, at a point v_{12} between v_1 and v_2 . The v_{12} possibility prompted our earlier speculation that we might have to deal with non-classical vertices.

The Hamiltonian probably has to allow for both possibilities, both the loop ending at v_2 , and the loop ending at v_{12} . If the loop only ends at points v_{12} , short of v_2 , then no disturbance can propagate along the lattice from v_1 to v_2 and beyond [7]. If the loop ends only at the next, already existing vertex, v_2 , then the number of vertices, n , is a good quantum number. This seems a bit too simple. For a given fixed total length, the Hamiltonian should be able to create more vertices, with less spin per vertex, if the gravitational self-interaction produces excitations of shorter wavelength.

If the Hamiltonian can create a new, spin 1/2 vertex, then we may have to deal with a highly non-classical action of the Hamiltonian, even in the classical limit. To investigate this possibility, I switch to a path integral point of view, and estimate the change in action corresponding to addition of a spin 1/2 loop at v_{12} .

I have to be careful which path integral I choose. In the introduction to paper 1, I discussed spin foams briefly. The spin foam approach is covariant, uses a path integral, and produces a satisfactory evolution operator; but it is not clear how to extract a canonical Hamiltonian from the operator [8, 9]. If I wish to do a "thought calculation" involving a path integral, I must stay on the canonical side of the canonical/covariant divide.

As pointed out in the previous section, the planar case has an asymptotic region, therefore a genuine Hamiltonian. I use this Hamiltonian (rather than the spin foam evolution operator) to construct a path integral.

This choice of Hamiltonian leads to an action which is order \sqrt{L} , like the Hamiltonian. Proof: the action and Hamiltonian differ by a term $(\tilde{E}/\kappa)\dot{A}d^3x$. This term is order \sqrt{L} :

$$\begin{aligned} (\tilde{E}/\kappa)\dot{A}d^3x &\sim (\tilde{E}/\kappa)dx^a dx^b h_c[H, h_c^{-1}]\epsilon_{abc} \\ &\sim L d(\sqrt{L/\kappa})/dL \\ &\sim L(1/\sqrt{L\kappa}). \end{aligned}$$

On the second line I have assumed that the commutator of h with the Hamiltonian is a derivative with respect to L . \square

This action must be exponentiated to give a path integral, and

the integration measure contains delta functions which enforce the constraints. I assume the integration is confined to the constraint surface; therefore I can ignore the ghost fields needed to exponentiate the constraints. (This is clearly a thought calculation, rather than an actual calculation!)

Since the action goes as \sqrt{L} , the change in the action, due to a change δL in spin, is

$$\delta S/S \sim \delta L/L.$$

The classical limit is attained by minimizing fluctuations of the action. The L in the denominator suggests that vertices involving small spins will not contribute significantly in the classical limit.

In the classical limit, it is plausible that results should approach those obtained from field theory. Presumably this requires that parameters u, \vec{p} vary slowly from vertex to vertex. (If one requires only that each individual vertex is coherent, this may not be enough to obtain a classical limit.) A requirement of slow variation would again rule out vertices with spin near $1/2$.

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